

HYPERKAHLER QUOTIENTS AND N=4 GAUGE THEORIES IN D=2 ¹

Marco Billó and Pietro Fré

*SISSA - International School for Advanced Studies, via Beirut 2-4, 34140 Trieste, Italy
and I.N.F.N. - Sezione di Trieste - Trieste, Italy*

1 Introduction

In this contribution we review some results we have recently obtained on the relationship between certain geometrical constructions of HyperKähler manifolds and N=4 supersymmetry in D=2 space-time. HyperKähler manifolds are of interest in connection with several issues in contemporary theoretical physics:

(i) non-compact HyperKähler manifolds in 4 dimensions can be interpreted as gravitational instantons;

(ii) 4-dimensional σ -models with compact or non-compact HyperKähler $4n$ -dimensional target manifolds describe the interaction of hypermultiplets in the N=2, D=4 supersymmetric theories.

In particular very recently the role (ii) played by HyperKähler manifolds has come under attention in relation with monopole theory, non-perturbative calculations in N=2 gauge theories and topological field theories in D=4 [1, 2, 3, 4, 5].

From the point of view (i) an important question not yet fully answered concerns gravitational instanton effects in string theory. Some progress has been made in constructing (4,4), $c=6$ superconformal models describing string propagation on such backgrounds [6, 7]; yet one has still to go a long way in order to master this kind of problems. Since string propagation on any manifold \mathcal{M} is described by a $2D$ σ -model with \mathcal{M} as target space it is clear that any $2D$ field theory whose effective theory is a σ -model on a HyperKähler \mathcal{M} has a deep meaning in the above context.

Here we shall discuss precisely this kind of theories. They are N=4 gauge theories coupled to hypermultiplets with canonical kinetic terms, the only interaction being induced by the minimal gauge coupling. Yet the structure of N=4 supersymmetry provides the link with the geometrical set-up of HyperKähler quotients. The basic result [8] is that the auxiliary fields of the gauge multiplet realize in a lagrangian way the triholomorphic momentum map and lead to an effective theory that has the HyperKähler quotient as target space. What we have found is the N=4 generalization of a pattern discovered by

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Witten in N=2,D=2 gauge theories. The essential difference is that while the N=2 case displays a two-phase structure, a σ -model phase and a Landau-Ginzburg phase, the N=4 case admits only the σ -model phase.

In this framework and using the geometrical results of Kronheimer [9, 10] (reviewed in this note) we show how to construct a *microscopic field theory* that admits as effective theory the N=4 σ -model on all Asymptotically Locally Euclidean (ALE) self-dual spaces. The physical data contained in the microscopic lagrangian (the Fayet-Iliopoulos parameters [11]) are put by our construction into *explicit* correspondence with the geometrical data associated with the ALE manifold, in particular the moduli of the complex structures. This correspondence is believed to be of relevance in the topological theories obtained by twisting either our microscopic two-dimensional theory [8] or N=2 supergravity in four dimensions [12, 13, 1].

2 HyperKähler quotients

HyperKähler manifolds

On a HyperKähler manifold \mathcal{M} , which is necessarily $4n$ -dimensional, there exist three covariantly constant complex structures $\mathcal{J}^i : T\mathcal{M} \rightarrow T\mathcal{M}$, $i = 1, 2, 3$; the metric is hermitean with respect to all of them and they satisfy the quaternionic algebra: $\mathcal{J}^i \mathcal{J}^j = -\delta^{ij} + \varepsilon^{ijk} \mathcal{J}^k$.

In a vierbein basis $\{V^A\}$, hermiticity of the metric is equivalent to the statement that the matrices \mathcal{J}_{AB}^i are antisymmetric. By covariant constancy, the three HyperKähler two-forms $\Omega^i = \mathcal{J}_{AB}^i V^A \wedge V^B$ are closed: $d\Omega^i = 0$. In the four-dimensional case, because of the quaternionic algebra constraint, the \mathcal{J}_{AB}^i can be either selfdual or antiselfdual; if we take them to be antiselfdual: $\mathcal{J}_{AB}^i = -\frac{1}{2}\varepsilon_{ABCD}\mathcal{J}_{CD}^i$, then the integrability condition for the covariant constancy of \mathcal{J}^i forces the curvature two-form R^{AB} to be selfdual: thus, in the four-dimensional case, HyperKähler manifolds are particular instances of gravitational instantons.

A HyperKähler manifold is a Kähler manifold with respect to each of its complex structures.

Momentum map

Consider a compact Lie group G acting on a HyperKähler manifold \mathcal{S} of real dimension $4n$ by means of Killing vector fields \mathbf{X} that are holomorphic with respect to the three complex structures of \mathcal{S} ; then these vector fields preserve also the HyperKähler forms:

$$\left. \begin{array}{l} \mathcal{L}_{\mathbf{X}}g = 0 \leftrightarrow \nabla_{(\mu}X_{\nu)} = 0 \\ \mathcal{L}_{\mathbf{X}}\mathcal{J}^i = 0, \quad i = 1, 2, 3 \end{array} \right\} \Rightarrow 0 = \mathcal{L}_{\mathbf{X}}\Omega^i = i_{\mathbf{X}}d\Omega^i + d(i_{\mathbf{X}}\Omega^i) = d(i_{\mathbf{X}}\Omega^i). \quad (1)$$

Here $\mathcal{L}_{\mathbf{X}}$ and $i_{\mathbf{X}}$ denote respectively the Lie derivative along the vector field \mathbf{X} and the contraction (of forms) with it.

If \mathcal{S} is simply connected, $d(i_{\mathbf{X}}\Omega^i) = 0$ implies the existence of three functions $\mathcal{D}_i^{\mathbf{X}}$ such that $d\mathcal{D}_i^{\mathbf{X}} = i_{\mathbf{X}}\Omega^i$. The functions $\mathcal{D}_i^{\mathbf{X}}$ are defined up to a constant, which can be arranged so to make them equivariant: $\mathbf{X}\mathcal{D}_i^{\mathbf{Y}} = \mathcal{D}_i^{[\mathbf{X}, \mathbf{Y}]}$.

The $\{\mathcal{D}_i^{\mathbf{X}}\}$ constitute then a *momentum map*. This can be regarded as a map $\mathcal{D} : \mathcal{S} \rightarrow \mathbf{R}^3 \otimes \mathcal{G}^*$, where \mathcal{G}^* denotes the dual of the Lie algebra \mathcal{G} of the group G . Indeed let $x \in \mathcal{G}$ be the Lie algebra element corresponding to the Killing vector \mathbf{X} ; then, for a given

$m \in \mathcal{S}$, $\mathcal{D}_i(m) : x \mapsto \mathcal{D}_i^{\mathbf{X}}(m) \in \mathbf{C}$ is a linear functional on \mathcal{G} . In practice, expanding $\mathbf{X} = X_a \mathbf{k}^a$ in a basis of Killing vectors \mathbf{k}^a such that $[\mathbf{k}^a, \mathbf{k}^b] = f^{abc} \mathbf{k}^c$, where f^{abc} are the structure constants of \mathcal{G} , we also have $\mathcal{D}_i^{\mathbf{X}} = X_a \mathcal{D}_i^a$, $i = 1, 2, 3$; the $\{\mathcal{D}_i^a\}$ are the components of the momentum map.

HyperKähler quotient

It is a procedure that provides a way to construct from \mathcal{S} a lower-dimensional HyperKähler manifold \mathcal{M} , as follows. Let $\mathcal{Z}^* \subset \mathcal{G}^*$ be the dual of the centre of \mathcal{G} . For each $\zeta \in \mathbf{R}^3 \otimes \mathcal{Z}^*$ the level set of the momentum map

$$\mathcal{N} \equiv \bigcap_i \mathcal{D}_i^{-1}(\zeta^i) \subset \mathcal{S}, \quad (2)$$

which has dimension $\dim \mathcal{N} = \dim \mathcal{S} - 3 \dim G$, is invariant under the action of G , due to the equivariance of \mathcal{D} . It is thus possible to take the quotient

$$\mathcal{M} = \mathcal{N}/G.$$

\mathcal{M} is a smooth manifold of dimension $\dim \mathcal{M} = \dim \mathcal{S} - 4 \dim G$ as long as the action of G on \mathcal{N} has no fixed points. The three two-forms ω^i on \mathcal{M} , defined via the restriction to $\mathcal{N} \subset \mathcal{S}$ of the Ω^i and the quotient projection from \mathcal{N} to \mathcal{M} , are closed and satisfy the quaternionic algebra thus providing \mathcal{M} with a HyperKähler structure.

Once \mathcal{J}^3 is chosen as the preferred complex structure, the momentum maps $\mathcal{D}_{\pm} = \mathcal{D}_1 \pm i\mathcal{D}_2$ are holomorphic (resp. antiholomorphic) functions.

The standard use of the HyperKähler quotient is that of obtaining non trivial HyperKähler manifolds starting from a flat $4n$ real-dimensional manifold \mathbf{R}^{4n} acted on by a suitable group G generating triholomorphic isometries [14, 15].

This is the way it was utilized by Kronheimer [9, 16] in its exhaustive construction of all self-dual asymptotically locally Euclidean four-spaces (ALE manifolds) [17, 10, 18, 19], that we consider later.

Indeed the manifold \mathbf{R}^{4n} can be given a quaternionic structure, and the corresponding quaternionic notation is sometimes convenient. For $n = 1$ one has the flat quaternionic space $\mathbf{H} \stackrel{\text{def}}{=} (\mathbf{R}^4, \{J^i\})$. We represent its elements

$$q \in \mathbf{H} = x + iy + jz + kt = x^0 + x^i J^i, \quad x, y, z, t \in \mathbf{R}$$

realizing the quaternionic structures J^i by means of Pauli matrices: $J^i = i(\sigma^i)^T$. Thus

$$q = \begin{pmatrix} u & iv^* \\ iv & u^* \end{pmatrix} \longrightarrow q^\dagger = \begin{pmatrix} u^* & -iv^* \\ -iv & u \end{pmatrix} \quad (3)$$

where $u = x^0 + ix^3$ and $v = x^1 + ix^2$. The euclidean metric on \mathbf{R}^4 is retrieved as $dq^\dagger \otimes dq = ds^2 \mathbf{1}$. The HyperKähler forms are grouped into a quaternionic two-form

$$\Theta = dq^\dagger \wedge dq \stackrel{\text{def}}{=} \Omega^i J^i = \begin{pmatrix} i\Omega^3 & i\Omega^+ \\ i\Omega^- & -i\Omega^3 \end{pmatrix}. \quad (4)$$

For generic n , we have the space \mathbf{H}^n , of elements

$$q = \begin{pmatrix} u^A & iv^{A*} \\ iv^A & u^{A*} \end{pmatrix} \longrightarrow q^\dagger = \begin{pmatrix} u^{A*} & -iv^{A*} \\ -iv^A & u^A \end{pmatrix} \quad \begin{matrix} u^A, v^A \in \mathbf{C}^n \\ A = 1, \dots, n \end{matrix} \quad (5)$$

Thus $dq^\dagger \otimes dq = ds^2 \mathbf{1}$ gives $ds^2 = du^{A*} \otimes du^A + dv^{A*} \otimes dv^A$ and the HyperKähler forms are grouped into the obvious generalization of the quaternionic two-form in eq.(4): $\Theta = \sum_{A=1}^n dq^{A\dagger} \wedge dq^A = \Omega^i J^i$, leading to $\Omega^3 = 2i\partial\bar{\partial}K$ where the Kähler potential K is $K = \frac{1}{2} (u^{A*} u^A + v^{A*} v^A)$, and to $\Omega^+ = 2idu^A \wedge dv^A$, $\Omega^- = (\Omega^+)^*$.

Let $(T_a)_B^A$ be the antihermitean generators of a compact Lie group G in its $n \times n$ representation. A triholomorphic action of G on \mathbf{H}^n is realized by the Killing vectors of components

$$X_a = (\hat{T}_a)_B^A q^B \frac{\partial}{\partial q^A} + q^{B\dagger} (\hat{T}_a)_A^B \frac{\partial}{\partial q^{A\dagger}} \quad ; \quad (\hat{T}_a)_B^A = \begin{pmatrix} (T_a)_B^A & \mathbf{0} \\ \mathbf{0} & (T_a^*)_B^A \end{pmatrix}. \quad (6)$$

Indeed one has $\mathcal{L}_X \Theta = 0$. The corresponding components of the momentum map are:

$$\mathcal{D}^a = q^{A\dagger} \begin{pmatrix} (T_a)_B^A & \mathbf{0} \\ \mathbf{0} & (T_a^*)_B^A \end{pmatrix} q^B + \begin{pmatrix} c & \bar{b} \\ b & -ic \end{pmatrix} \quad (7)$$

where $c \in \mathbf{R}$, $b \in \mathbf{C}$ are constants.

3 HyperKähler quotients in D=2, N=4 theories

Consider a supersymmetric σ -model from a 2-dimensional N=4 super-world sheet to a target space \mathcal{S} . It is well known that supersymmetry requires \mathcal{S} to be HyperKähler. Introduce also the supersymmetric gauge multiplet for a group G acting triholomorphically on \mathcal{S} . It turns out that the auxiliary fields of the gauge multiplet $\{\mathcal{P}, \mathcal{Q}\}$, ($\mathcal{P} \in \mathbf{R}$, $\mathcal{Q} \in \mathbf{C}$) are identified with the momentum functions $\{\mathcal{D}^3, \mathcal{D}^\pm\}$ for the G -action on \mathcal{S} .

In view of this fundamental property, the HyperKähler quotient offers a natural way to construct a N=4, D=2 σ -model on a non-trivial manifold \mathcal{M} starting from a free σ -model on a flat-manifold $\mathcal{S} = \mathbf{H}^n$. It suffices to gauge appropriate triholomorphic isometries by means of non-propagating gauge multiplets. Omitting the kinetic term of these gauge multiplets and performing the gaussian integration of the corresponding fields one realizes the HyperKähler quotient in a Lagrangian way.

Actually the HyperKähler quotient is a generalization of a similar Kähler quotient procedure, where the momentum map $\mathcal{D} : \mathcal{S} \rightarrow \mathbf{R} \otimes \mathcal{G}^*$ consists just of one hamiltonian function, rather than three. The Kähler quotient is related with N=2, D=2 supersymmetry, the reason being that in this case the vector multiplet contains just one real auxiliary field \mathcal{P} .

Recently, Witten has reconsidered the Kähler quotient construction of an N=2 two-dimensional σ -model in [20]. We constructed in [8] the analogue N=4 model, and compared it with the N=2 model. We review here some fundamental points of this work.

Results for the N=2 theory

In the N=2 case a vector multiplet is composed of a gauge boson \mathcal{A} , namely a world-sheet 1-form, two spin 1/2 gauginos, whose four components we denote by $\lambda^+, \lambda^-, \tilde{\lambda}^+, \tilde{\lambda}^-$, a complex physical scalar M and a real auxiliary scalar \mathcal{P} . For simplicity we write here the formulæ for a U(1) gauge group² and we consider a linear superpotential [20]. $i\frac{t}{4}M$,

²The extension to several abelian groups is trivial and for a generic group it is contained in [8] in the same setting as here.

with the coupling t is $t = ir + \frac{\theta}{2\pi}$. Denote $\partial_{\pm} = \partial/\partial z^0 \pm \partial/\partial z^1$, z^{α} being the world-sheet coordinates. Then we have:

$$\begin{aligned}\mathcal{L}_{\text{gauge}}^{(2)} &= \frac{1}{2}\mathcal{F}^2 - i(\tilde{\lambda}^+ \partial_+ \tilde{\lambda}^- + \lambda^+ \partial_- \lambda^-) - 4(\partial_+ M^* \partial_- M + \partial_- M^* \partial_+ M) \\ &\quad + 2\mathcal{P}^2 - 2r\mathcal{P} + \frac{\theta}{2\pi}\mathcal{F}\end{aligned}\quad (8)$$

$r\mathcal{P}$ is the Fayet-Iliopoulos term [11], $\theta\mathcal{F}/2\pi$ a topological term. We consider then n chiral multiplets $X^i, \psi^i, \tilde{\psi}^i, \mathcal{H}_i$ (plus their complex conjugates) of charge q^i with respect to the above $U(1)$ group. The complex scalars X^i span \mathbf{C}^n . The complex auxiliary field \mathcal{H}^i is identified with the derivative of a holomorphic superpotential $W(X)$: $\mathcal{H}^i = \partial_{i^*} W^*$. The supersymmetric lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{chiral}}^{(2)} &= -(\nabla_+ X^{i^*} \nabla_- X^i + \nabla_- X^{i^*} \nabla_+ X^i) + 4i(\psi^i \nabla_- \psi^{i^*} + \tilde{\psi}^i \nabla_+ \tilde{\psi}^{i^*}) \\ &\quad + 8\left((\psi^i \tilde{\psi}^j \partial_i \partial_j W + \text{c.c.}) + \partial_i W \partial_{i^*} W^*\right) + 2i \sum_i q^i (\psi^i \tilde{\lambda}^- X^{i^*} - \tilde{\psi}^i \lambda^- X^{i^*} - \text{c.c.}) \\ &\quad + 8i\left(M^* \sum_i q^i \psi^i \tilde{\psi}^{i^*} - \text{c.c.}\right) + 8M^* M \sum_i (q^i)^2 X^{i^*} X^i - 2\mathcal{P} \sum_i q^i X^{i^*} X^i\end{aligned}\quad (9)$$

Here and henceforth ∇ is the covariant derivative constructed by means of the gauge connection \mathcal{A} .

The total lagrangian is $\mathcal{L}^{(2)} = \mathcal{L}_{\text{gauge}}^{(2)} - \mathcal{L}_{\text{chiral}}^{(2)}$, the relative sign being fixed by the requirement of positivity of the energy. Note that $\mathcal{D}^{\mathbf{X}}(X, X^*) = \sum_i q^i |X^i|^2$ is the momentum map function for the holomorphic action of the gauge group on the matter multiplets. Eliminating \mathcal{P} through its own equation of motion we get: $\mathcal{P} = -\frac{1}{2}(D^{\mathbf{X}}(X, X^*) - r)$ and the bosonic potential reduces to:

$$U = \frac{1}{2}\left[r - \sum_i q^i |X^i|^2\right]^2 + 8|\partial_i W|^2 + 8|M|^2 \sum_i (q^i)^2 |X^i|^2 \quad (10)$$

Let us focus now on the low energy effective theory for this model. First of all consider the structure of the classical vacua. We must extremize the potential (10). Witten considered [20] the case with L.G. potential $W = X^0 \mathcal{W}(X^i)$ where $\mathcal{W}(X^i)$ is quasi homogeneous of degree d if the X^i 's are assigned as homogeneity weights their charges q^i , and is transverse: $\partial_i \mathcal{W} = 0 \forall i$ iff $X^i = 0 \forall i$. X^0 has charge $-d$. Then two phases appear:

- $r > 0$: *σ -model phase*. The space of classical vacua is a transverse hypersurface embedded in $\mathbf{WCP}_{q^1 \dots q^N}^N$: $X^0 = M = 0$, $\sum_i q^i |X^i|^2 = r$, $\mathcal{W}(X^i) = 0$.
- $r < 0$: *Landau-Ginzburg phase*. The space of vacua is a point: $X^0 = \sqrt{\frac{-r}{d}}$, $X^i = M = 0$. The low-energy theory describes massless fields governed by a Landau-Ginzburg potential $\mathcal{W}(X^i)$.

Our interest is in the low-energy theory around a vacuum of the first type. This theory turns out to be correctly described by a $N=2$ σ -model [8]. It emerges via the lagrangian realization of a Kähler quotient. Here we look just at the bosonic fields. To be simple and definite, consider the \mathbf{CP}^n model, that is the case $q^A = 1$, $A = 0, 1, \dots, n$ and $W(X^A) = 0$. In this case there is only the σ -model phase.

Reinstall the gauge coupling constant g in the lagrangian $\mathcal{L}^{(2)}$. Then let $g \rightarrow \infty$. We are left with a gauge invariant lagrangian describing matter coupled to gauge fields that have no kinetic terms. Then we vary the action in these fields. The resulting equations of motion express the gauge fields in terms of the matter fields. This procedure is nothing else, from the functional integral viewpoint, but the gaussian integration over the gauge multiplet in the limit $g \rightarrow \infty$. It amounts to deriving the low-energy effective action around the classical vacua of the complete, gauge plus matter system. Indeed we have seen that around these vacua the oscillations of the gauge fields are massive, and thus decouple from the low-energy point of view.

In our example, the variation in the gauge connection components \mathcal{A}_\pm and in M identifies them in terms of the matter fields. In particular $\mathcal{A}_\pm = -i(X\partial_\pm X^* - X^*\partial_\pm X)/2X^*X$. This has to be substituted into the lagrangian. This latter is by construction invariant under the $U(1)$ transformation $X^A \rightarrow e^{i\Phi}X^A$, $\Phi \in \mathbf{R}$. Allow now $\Phi \in \mathbf{C}$, thus complexifying $U(1)$ to $\mathbf{C}^* = \mathbf{C} - \{0\}$. Introduce an extra field v , transforming under \mathbf{C}^* as $v \rightarrow v + \frac{i}{2}(\Phi - \Phi^*)$. Then the combinations $e^{-v}X^A$ undergo just a $U(1)$ transformation: $e^{-v}X^A \rightarrow e^{i\text{Re}\Phi}e^{-v}X^A$.

By substituting in the lagrangian $X^A \rightarrow e^{-v}X^A$ it becomes \mathbf{C}^* -invariant. In particular the term involving \mathcal{P} becomes $-2\mathcal{P}(r - e^{-2v}X^*X)$. Performing the variation with respect to the auxiliary field \mathcal{P} , the resulting equation of motion identifies the extra scalar field v in terms of the matter fields. Introducing $\rho^2 \equiv r$ the result is that $e^{-v} = \frac{\rho}{\sqrt{X^*X}}$.

What is the geometrical meaning of the above “tricks” (introduction of the extra field v , consideration of the complexified gauge group)? The answer relies on the properties of the Kähler quotient construction [14]; Let us recall a few concepts, using notions and notations introduced in section 1

Let $\mathbf{Y}(s) = Y^a \mathbf{k}_a(s)$ be a Killing vector on \mathcal{S} (in our case \mathbf{C}^{N+1}), belonging to \mathcal{G} (in our case \mathbf{R}), the algebra of the gauge group. In our case \mathbf{Y} has a single component: $\mathbf{Y} = i\Phi(X^A \frac{\partial}{\partial X^A} - X^{A*} \frac{\partial}{\partial X^{A*}})$ ($\Phi \in \mathbf{R}$). The X^A 's are the coordinates on \mathcal{S} . Consider the vector field $I\mathbf{Y} \in \mathcal{G}^c$ (the complexified algebra), I being the complex structure acting on $T\mathcal{S}$. In our case $I\mathbf{Y} = \Phi(X^A \frac{\partial}{\partial X^A} + X^{A*} \frac{\partial}{\partial X^{A*}})$. This vector field is orthogonal to the hypersurface $\mathcal{D}^{-1}(\zeta)$, for any level ζ ; that is, it generates transformations that change the level of the surface. In our case the surface $\mathcal{D}^{-1}(\rho^2) \in \mathbf{C}^{N+1}$ is defined by the equation $X^{A*}X^A = \rho^2$. The infinitesimal transformation generated by $I\mathbf{Y}$ is $X^A \rightarrow (1 + \Phi)X^A$, $X^{A*} \rightarrow (1 + \Phi)X^{A*}$ so that the transformed X^A 's satisfy $X^{A*}X^A = (1 + 2\Phi)\rho^2$. As recalled in section I, the Kähler quotient consists in starting from \mathcal{S} , restricting to $\mathcal{N} = \mathcal{D}^{-1}(\zeta)$ and taking the quotient $\mathcal{M} = \mathcal{N}/G$. The above remarks about the action of the complexified gauge group suggest that this is equivalent (at least if we skip the problems due to the non-compactness of G^c) to simply taking the quotient \mathcal{S}/G^c , the so-called “algebraic-geometric” quotient [14, 21].

The Kähler quotient allows in principle to determine the expression of the Kähler form on \mathcal{M} in terms of the original one on \mathcal{S} . Schematically, let j be the inclusion map of \mathcal{N} into \mathcal{S} , p the projection from \mathcal{N} to the quotient $\mathcal{M} = \mathcal{N}/G$, Ω the Kähler form on \mathcal{S} and ω the Kähler form on \mathcal{M} . It can be shown [14] that

$$\begin{array}{ccccc} \mathcal{S} & \xleftarrow{j} & \mathcal{N} = \mathcal{D}^{-1}(\zeta) & \xrightarrow{p} & \mathcal{M} = \mathcal{N}/G \\ \Omega & \longrightarrow & j^*\Omega = p^*\omega & \longleftarrow & \omega \end{array} \quad (11)$$

In the algebro-geometric setting, the holomorphic map that associates to a point $s \in \mathcal{S}$ (for us, $\{X^A\} \in \mathbf{C}^{N+1}$) its image $m \in \mathcal{M}$ is obtained as follows:

i) Bringing s to \mathcal{N} by means of the finite action infinitesimally generated by a vector field of the form $\mathbf{V} = I\mathbf{Y} = V^a \mathbf{k}_a$

$$\pi : s \in \mathcal{S} \longrightarrow e^{-V} s \in \mathcal{D}^{-1}(\zeta) \quad (12)$$

ii) Projecting e^{-V} to its image in the quotient $\mathcal{M} = \mathcal{N}/G$.

Thus we can consider the pullback of the Kähler form ω through the map $p \cdot \pi$:

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{\pi} & \mathcal{N} = \mathcal{D}^{-1}(\zeta) & \xrightarrow{p} & \mathcal{N}/G \\ \pi^* p^* \omega & \longleftarrow & p^* \omega & \longleftarrow & \omega \end{array} \quad (13)$$

Looking at (11) we see that $\pi^* p^* \omega = \pi^* j^* \Omega$ so that at the end of the day, in order to recover the pullback of ω to \mathcal{S} it is sufficient:

i) to restrict Ω to \mathcal{N}

ii) to pull back this restriction to \mathcal{M} with respect to the map $\pi = e^{-V}$.

We see from (12) that the components of the vector field \mathbf{V} must be determined by requiring

$$\mathcal{D}(e^{-V} s) = \zeta \quad (14)$$

In the lagrangian context, after having introduced the extra field v (which is now interpreted as the unique component of the vector field \mathbf{V}) to make the lagrangian \mathbf{C}^* -invariant, eq. (14) is retrieved as the equation of motion for \mathcal{P} . We have determined the form of the map π : it corresponds on the bosonic fields to $X^A \rightarrow e^{-v} X^A$.

The remaining steps in treating the lagrangian just consist in implementing the Kähler quotient as in (13). At the end we obtain the σ -model on the target space \mathcal{M} (in our case \mathbf{CP}^N) endowed with the Kähler metric corresponding to the Kähler form ω . In our example such metric is the Fubini-Study metric. Indeed in full generality one can show [14] that the Kähler potential \hat{K} for the manifold \mathcal{M} , such that $\omega = 2i\partial\bar{\partial}\hat{K}$ is given by

$$\hat{K} = K|_{\mathcal{N}} + V^a \zeta_a \quad (15)$$

Here K is the Kähler potential on \mathcal{S} ; $K|_{\mathcal{N}}$ is its restriction to \mathcal{N} , that is, it is computed after acting on the point $s \in \mathcal{S}$ with the transformation e^{-V} determined by eq. (14); V^a are the components of the vector field \mathbf{V} along the a^{th} generator of the gauge group, and ζ_a those of the level ζ of the momentum map; recall that the ζ belong to the dual of the center, \mathcal{Z}^* and therefore only the components of \mathbf{V} along the center actually contribute to eq. (15). In our case we have the single component v given by $e^{-v} = \rho^2 / \sqrt{X^* X}$; ρ^2 is the single component of the level. The original Kähler potential on $\mathcal{S} = \mathbf{C}^{N+1}$ is $K = \frac{1}{2} X^{A*} X^A$ so that when restricted to $\mathcal{D}^{-1}(\rho^2)$ it takes an irrelevant constant value $\frac{\rho^2}{2}$. Thus we deduce from (15) that the Kähler potential for $\mathcal{M} = \mathbf{CP}^N$ that we obtain is $\hat{K} = \frac{1}{2} \rho^2 \log(X^* X)$. Fixing a particular gauge to perform the quotient with respect to \mathbf{C}^* (see later), this potential can be rewritten as $\hat{K} = \frac{1}{2} \rho^2 \log(1 + x^* x)$, namely the Fubini-Study potential.

Indeed it is trivial to rewrite the lagrangian after the substitution $X^A \rightarrow (\rho^2 / \sqrt{X^* X}) X^A$. We can then utilize the gauge invariance to fix for instance, in the coordinate patch where

$X^0 \neq 0$, $X^0 = 1$. That is, we fix completely the gauge going from the homogeneous coordinates (X^0, X^i) to the inhomogeneous coordinates $(1, x^i = X^i/X^0)$ on \mathbf{CP}^N .

Having chosen our gauge, we rewrite the lagrangian in terms of the fields x^i . Thus the bosonic lagrangian reduces finally to that of a σ -model on \mathbf{CP}^n :

$$\mathcal{L} = g_{ij^*}(\partial_+ x^i \partial_- x^{j^*} + \partial_- x^i \partial_+ x^{j^*}) \quad (16)$$

where g_{ij^*} is the Fubini-Study metric, $g_{ij^*} = \frac{\rho^2}{1+x^*x} \left(\delta_{ij} - \frac{x^{i^*}x^j}{1+x^*x} \right)$.

The N=4 theory

The N=4 vector multiplet³, in addition to the gauge boson, namely the 1-form \mathcal{A} , contains four spin 1/2 gauginos whose eight components are denoted by $\lambda^+, \lambda^-, \tilde{\lambda}^+, \tilde{\lambda}^-, \mu^+, \mu^-, \tilde{\mu}^+, \tilde{\mu}^-$, two complex physical scalars $M \neq M^*$, $N \neq N^*$, and three auxiliary fields arranged into a real scalar $\mathcal{P} = \mathcal{P}^*$ and a complex scalar $\mathcal{Q} \neq \mathcal{Q}^*$.

$$\begin{aligned} \mathcal{L}_{\text{gauge}}^{(4)} = & \frac{1}{2} \mathcal{F}^2 - i(\tilde{\lambda}^+ \partial_+ \tilde{\lambda}^- + \tilde{\mu}^+ \partial_+ \mu^- + \lambda^+ \partial_- \lambda^- + \mu^+ \partial_- \tilde{\mu}^-) \\ & + 4(\partial_+ M^* \partial_- M + \partial_- M^* \partial_+ M + \partial_+ N^* \partial_- N + \partial_- N^* \partial_+ N) \\ & + \frac{\theta}{2\pi} \mathcal{F} + 2\mathcal{P}^2 + 2\mathcal{Q}^* \mathcal{Q} - 2r\mathcal{P} - (s\mathcal{Q}^* + s^*\mathcal{Q}) \end{aligned} \quad (17)$$

Note that in addition to the Fayet-Iliopoulos and the topological term present in the N=2 case we have term linear in the complex auxiliary field \mathcal{Q} , involving a new complex parameter s .

The quaternionic hypermultiplets are the N=4 analogues of the N=2 chiral multiplets. They are described by a set of bosonic complex fields u^i, v^i , that can be organized in quaternions

$$Y^i = \begin{pmatrix} u^i & i v^{i^*} \\ i v^i & u^{i^*} \end{pmatrix} \quad (18)$$

spanning \mathbf{H}^n . Their supersymmetric partners are four spin 1/2 fermions, whose eight components we denote by $\psi_u^i, \tilde{\psi}_u^i, \psi_v^i, \tilde{\psi}_v^i$ together with their complex conjugates $\psi_u^{i^*}, \tilde{\psi}_u^{i^*}, \psi_v^{i^*}, \tilde{\psi}_v^{i^*}$. On these matter fields the abelian gauge group acts in a *triholomorphic* fashion, which in our setting means that $\{u^i, v^i\}$ have charges $\{q^i, -q^i\}$. The lagrangian is:

$$\begin{aligned} \mathcal{L}_{\text{quatern}}^{(4)} = & -(\nabla_+ u^{i^*} \nabla_- u^i + \nabla_- u^{i^*} \nabla_+ u^i + \nabla_+ v^{i^*} \nabla_- v^i + \nabla_- v^{i^*} \nabla_+ v^i) \\ & + 4i(\psi_u^i \nabla_- \psi_u^{i^*} + \psi_v^i \nabla_- \psi_v^{i^*} + \tilde{\psi}_u^i \nabla_+ \tilde{\psi}_u^{i^*} + \tilde{\psi}_v^i \nabla_+ \tilde{\psi}_v^{i^*}) \\ & + 2i \sum_i q^i \left\{ \left[\psi_u^i (\tilde{\lambda}^- u^{i^*} + \tilde{\mu}^+ v^i) - \text{c.c.} \right] - \left[\psi_v^i (\tilde{\lambda}^- v^{i^*} - \tilde{\mu}^+ u^i) - \text{c.c.} \right] \right. \\ & \left. - \left[\tilde{\psi}_u^i (\lambda^- u^{i^*} + \mu^+ v^i) - \text{c.c.} \right] + \left[\tilde{\psi}_v^i (\lambda^- v^{i^*} - \mu^+ u^i) - \text{c.c.} \right] \right\} \\ & + 8i \left[M^* \sum_i q^i (\psi_u^i \tilde{\psi}_u^{i^*} - \psi_v^i \tilde{\psi}_v^{i^*}) - \text{c.c.} \right] - 8i \left[N \sum_i q^i (\psi_u^i \tilde{\psi}_v^i + \psi_v^i \tilde{\psi}_u^i) - \text{c.c.} \right] \\ & + 8(|M|^2 + |N|^2) \sum_i (q^i)^2 (|u^i|^2 + |v^i|^2) - 2\mathcal{P} \sum_i q^i (|u^i|^2 - |v^i|^2) \\ & + 2i(\mathcal{Q} \sum_i q^i u^i v^i - \text{c.c.}) \end{aligned} \quad (19)$$

³As for N=2 we write the formulae for a U(1) gauge group and we refer for extension to a group G to [8].

Comparing with formulae (7) we see that the auxiliary field \mathcal{P} multiplies the real component $\mathcal{D}^3(u^i, v^i) = \sum_i q^i (|u^i|^2 - |v^i|^2)$, while \mathcal{Q} multiplies the holomorphic component $\mathcal{D}^-(u^i, v^i) = -2i \sum_i q^i u^i v^i$ of the momentum map.

Consider $\mathcal{L}^{(4)} = \mathcal{L}_{\text{gauge}}^{(4)} - \mathcal{L}_{\text{quatern}}^{(4)}$. Varying in \mathcal{P} and \mathcal{Q} we obtain :

$$\begin{aligned}\mathcal{P} &= \frac{1}{2} \left[r - \sum_i q^i (|u^i|^2 - |v^i|^2) \right] = \frac{1}{2} \left[r - \mathcal{D}^3(u, v) \right] \\ \mathcal{Q} &= \frac{1}{2} \left[s - 2i \sum_i q^i u^i v^i \right] = \frac{1}{2} \left[s - \mathcal{D}^+(u, v) \right]\end{aligned}\quad (20)$$

Using eq.s (20) from the lagrangian we can extract the final form of the N=4 bosonic potential. We make an exception to our choice of focusing on U(1) gauge group and write this potential in the case where the gauge group is the direct sum of several U(1)'s. This is done in order to make contact with the HyperKähler quotients of the next section. Use an index a to enumerate such U(1) factors; the triholomorphic action is described by the matrices $(F^a)_j^i$ acting on the u^j and $-F^a$ on the v^j fields. In practice one has to replace $q^i \rightarrow (F^a)_j^i$ and sum over a . Then we get

$$U = \sum_a \left(\frac{1}{2} (r_a - \mathcal{D}_a^3)^2 + \frac{1}{2} |s_a - \mathcal{D}_a^+|^2 + 8(|M_a|^2 + |N_a|^2) \sum_{i,j} (F^a F^a)_{ij} (u^{i*} u^j + v^{i*} v^j) \right) \quad (21)$$

As we see, the parameters r, s of the N=4 Fayet-Iliopoulos term are identified with the levels of the triholomorphic momentum-map.

Minimizing the potential (21) we find only a N=4 σ -model phase; the reason is the absence of an N=4 analogue of the Landau-Ginzburg potential. Besides $M = N = 0$, we must impose $\mathcal{D}^3(u, v) = r$ and $\mathcal{D}^+ = s$. Taking into account the gauge invariance of the Lagrangian, this means that the classical vacua are characterized by having $M = N = 0$ and the matter fields u, v lying on the HyperKähler quotient

$$\mathcal{M} = \mathcal{D}_3^{-1}(r) \cap \mathcal{D}_+^{-1}(s) / U(1) \quad (22)$$

of the quaternionic space \mathbf{H}^n spanned by the fields u^i, v^i with respect to the triholomorphic action of the $U(1)$ gauge group. Considering the fluctuations around this vacuum, we can see that the fields of the gauge multiplet are massive, together with the modes of the matter fields not tangent to \mathcal{M} . The low-energy theory will turn out to be the N=4 σ -model on \mathcal{M} .

It is useful at this point, in order to compare with the N=2 case, to note that N=4 theories are nothing else but particular N=2 theories whose structure allows the existence of additional supersymmetries.

Thus if we look at the N=4 theory described above from an N=2 point of view, it contains one gauge multiplet and $2n + 1$ chiral multiplets, whose bosonic components we denote as $\{X^A\}$, $A = 0, \dots, n$. They are explicitly $\{X^0 = 2N, u^i, v^i\}$, with charges $\{0, q^i, -q^i\}$. The Landau Ginzburg potential is

$$W(X^A) = -\frac{1}{4} X^0 (s^* - \mathcal{D}^-(u, v)) = -\frac{1}{4} X^0 \left(s^* + 2i \sum_i q^i u^i v^i \right) \quad (23)$$

where $\mathcal{D}^-(u, v) = -2i \sum_i q^i u^i v^i$ is the holomorphic part of the momentum map for the triholomorphic action of the gauge group on $\mathbf{H}^n \equiv \mathbf{C}^{2n}$. The Landau-Ginzburg potential being given by eq. (23), the form (10) of the N=2 bosonic potential reduces exactly to the potential of eq. (21) (for U(1) gauge group):

$$\begin{aligned} U &= \frac{1}{2} (r - \mathcal{D}^3)^2 + \sum_A |\partial_A W|^2 + 8|M|^2 \sum_i (q^i)^2 (|u^i|^2 + |v^i|^2) \\ &= \frac{1}{2} (r - \mathcal{D}^3)^2 + \frac{1}{2} |s - \mathcal{D}^+|^2 + (8|M|^2 + 2|X^0|^2) \sum_i (q^i)^2 (|u^i|^2 + |v^i|^2) \end{aligned} \quad (24)$$

From this N=2 point of view we do not see two different phases in the structure of the classical vacuum because of the expression of $\mathcal{D}^3(u, v)$, see eq. (20). It is indeed clear that by setting

$$r - \sum_i q^i (|u^i|^2 - |v^i|^2) = 0 \quad (25)$$

the exchange of $r > 0$ with $r < 0$ just corresponds to the exchange of the u 's with the v 's. Everywhere else the u 's and the v 's appear symmetrically, hence the two phases $r > 0$ and $r < 0$ are actually the same thing. This is far from being accidental. The charge of v^i is opposite to the one of u^i because of the triholomorphicity of the action of the gauge group, which is essential in a N=4 theory; thus the indistinguishability of the two phases is intrinsic to any N=4 theory of the type we are considering in this paper.

To complete the definition of the vacuum, we must set $M = X^0 = 0$ and require $\mathcal{D}^+(u, v) = s$.

We now examine the reconstruction of the low-energy theory. We again focus on the bosonic sector. We utilize the above N=2 point of view, and proceed as before. We just insist now on the peculiarities due to the form of the potential (24).

By letting the gauge coupling constant go to infinity, we eliminate the gauge kinetic terms and also that for X^0 , since it originates from the N=4 gauge multiplet.

Note that the holomorphic constraint $\mathcal{D}^+ = s$ is not implemented in the N=2 lagrangian we are starting from through a Lagrange multiplier. This would be the case (by means of the auxiliary field \mathcal{Q}) had we chosen to utilize the N=4 formalism, see eq.(19), and this is the case for the real constraint $\mathcal{D}^3 = r$, through the auxiliary field \mathcal{P} . This fact causes no problem, as it is perfectly consistent with what happens, from the geometrical point of view, taking the HyperKähler quotient. Indeed the HyperKähler quotient procedure is schematically represented by

$$\mathcal{S} \xleftarrow{j^+} \mathcal{D}_+^{-1}(s) \xleftarrow{j^3} \mathcal{N} \equiv \mathcal{D}_3^{-1}(r) \cap \mathcal{D}_+^{-1}(s) \xrightarrow{p} \mathcal{M} \equiv \mathcal{N}/G \quad (26)$$

where we have extended in an obvious way the notation of eq. (11): j^+ and j^3 are the inclusion maps and p the projection on the quotient.

We already remarked that the surface $\mathcal{D}_3^{-1}(r)$ is not invariant under the action of the *complexified* gauge group G^c . Instead it is easy to verify that the holomorphic surface $\mathcal{D}_+^{-1}(s)$ is invariant under the action of G^c . Just as in the Kähler quotient procedure we can therefore replace the restriction to $\mathcal{D}_3^{-1}(r)$ and the G quotient with a G^c quotient, without modifying the need of taking the restriction to $\mathcal{D}_+^{-1}(s)$. The HyperKähler quotient can be realized as follows:

$$\mathcal{S} \xleftarrow{j^+} \mathcal{D}_+^{-1}(s) \xrightarrow{p^c} \mathcal{M} \equiv \mathcal{D}_+^{-1}(s)/G^c \quad (27)$$

We see that, in any case, we have to implement the constraint $\mathcal{D}^+ = s$. This does not affect the procedure of extending the action of the gauge group to its complexification. Setting now, to be simple and definite, $q^i = 1 \forall i$ ⁴, the complexified group acts as:

$$\begin{aligned} u^i &\longrightarrow e^{i\Phi} u^i & ; & & v^i &\longrightarrow e^{-i\Phi} v^i \\ v &\longrightarrow v + \frac{i}{2}(\Phi - \Phi^*) \end{aligned} \quad (28)$$

One obtains the invariance of the lagrangian under this action by means of the substitutions $u^i \rightarrow e^{-v} u^i$, $v^i \rightarrow e^{-v} v^i$. The variation, after these replacements, in the auxiliary field \mathcal{P} gives the equation $\mathcal{D}^3(e^{-v}u, e^v v) = r$, that is

$$r - e^{-2v} \sum_i |u^i|^2 + e^{2v} \sum_i |v^i|^2 = 0 \quad (29)$$

This is easily solved for v . We have still to implement the holomorphic constraint $\mathcal{D}_+ = s$. This task is simplified by the \mathbf{C}^* gauge invariance of our lagrangian. As it is clear from the form of the \mathbf{C}^* -transformations one can for instance choose the gauge $u^n = v^n$. Solving explicitly the constraint gives the $\{u^i, v^i\}$, $i = 1, \dots, n$ in terms of some irreducible coordinates $\{\hat{u}^I, \hat{v}^I\}$, $I = 1, \dots, n-1$. The final result of the appropriate manipulations that should be made on the lagrangian will be the reconstruction of the action for the N=2 σ -model having as target space the HyperKähler quotient $\mathbf{H}^n/U(1)$, endowed with the Kähler metric which is naturally provided by this construction, exactly as it happened in the Kähler quotient case. The Kähler quotient is again obtained through eq. (15). It is convenient to call $\beta = \sum_i |u^i|^2$ and $\gamma = \sum_i |v^i|^2$ (both to be considered as functions of \hat{u}^I, \hat{v}^I). Differently from the \mathbf{CP}^N case, the part of the target space Kähler potential coming from the restriction of the Kähler potential for \mathbf{H}^n to the surface $\mathcal{D}_3^{-1}(r) \cap \mathcal{D}_+^{-1}(s)$ is not an irrelevant constant. Indeed it is given (see section 1) by:

$$K|_{\mathcal{N}} = \frac{1}{2}(e^{-2v} \sum_i |u^i|^2 + e^{2v} \sum_i |v^i|^2) = \frac{1}{2}\sqrt{\rho^4 + 4\beta\gamma} \quad (30)$$

The final expression of the Kähler potential for the Calabi metric is:

$$\hat{K} = \frac{1}{2}\sqrt{\rho^4 + 4\beta\gamma} + \frac{\rho^2}{2} \log \frac{-\rho^2 + \sqrt{\rho^4 + 4\beta\gamma}}{2\gamma} \quad (31)$$

In the case $n = 2$, the target space has 4 real dimensions and the Calabi metric is nothing else than the Eguchi-Hanson metric, i.e. the simplest Asymptotically Locally Euclidean (ALE) gravitational instanton [17, 14, 9].

4 ALE spaces as hyperKähler quotients

The most natural gravitational analogues of the Yang-Mills instantons are geodesically complete Riemannian four-manifolds with (anti)selfdual curvature 2-form⁵. One would

⁴This corresponds to the obvious N=4 generalization of \mathbf{CP}^N . The spaces obtained by means of the hyperKähler quotient procedure of \mathbf{H}^n with respect to this $U(1)$ action have real dimension $4(n-1)$; the Kähler metric they inherit from the quotient construction are called Calabi metrics [22]

⁵In 4 dimensions (anti)selfduality of the curvature implies that the spaces are HyperKähler and that their metric automatically satisfies the vacuum Einstein equations

like their metric to approach the euclidean metric at infinity, in agreement with the “intuitive” picture of instantons as being localized in finite regions of space-time. This behaviour is however possible only modulo an additional subtlety: the base manifold has a boundary at infinity S^3/Γ , Γ being a finite group of identifications. “Outside the core of the instanton” the manifold looks like \mathbf{R}^4/Γ instead of \mathbf{R}^4 . This is the reason of the name given to these spaces: the asymptotic behaviour is only *locally* euclidean. The unique *globally* euclidean gravitational instanton is euclidean four-space itself, which has boundary S^3 .

The simplest of the ALE metrics is the Eguchi-Hanson metric [17], which corresponds to the case where the boundary infinity is S^3/\mathbf{Z}_2 . The so-called Multi-center metrics [10, 19] correspond to the cases S^3/\mathbf{Z}_n . The general picture [19, 9] is as follows: every ALE space is determined by its group of identifications Γ , which must be a finite Kleinian subgroup of $SU(2)$. Kronheimer described indeed manifolds having such a boundary; he showed that in principle a unique selfdual metric can be obtained for each of these manifolds [9] and, moreover, that every selfdual metric approaching asymptotically the euclidean one can be recovered in such a manner [16].

The Kleinian subgroups of $SU(2)$

Choosing complex coordinates $z_1 = x - iy, z_2 = t + iz$ on $\mathbf{R}^4 \sim \mathbf{C}^2$, and representing a point (z_1, z_2) by a quaternion, the group $SO(4) \sim SU(2)_L \times SU(2)_R$, which is the isometry group of the sphere at infinity, acts on the quaternion by matrix multiplication:

$$\begin{pmatrix} z_1 & i\bar{z}_2 \\ iz_2 & \bar{z}_1 \end{pmatrix} \longrightarrow M_1 \cdot \begin{pmatrix} z_1 & i\bar{z}_2 \\ iz_2 & \bar{z}_1 \end{pmatrix} \cdot M_2 \quad (32)$$

the element $M \in SO(4)$ being represented as $(M_1 \in SU(2)_L, M_2 \in SU(2)_R)$. The group Γ can be seen as a finite subgroup of $SU(2)_L$, acting on \mathbf{C}^2 in the natural way by its two-dimensional representation:

$$\forall U \in \Gamma \subset SU(2), \quad U : \mathbf{v} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longrightarrow U\mathbf{v} = \begin{pmatrix} \alpha & i\beta \\ i\bar{\beta} & \bar{\alpha} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \quad (33)$$

It is a classic result that the possible finite subgroups of $SU(2)$ are organized in two infinite series and three exceptional cases; each subgroup Γ is in correspondence with a simply laced Lie algebra \mathcal{G} , and we write $\Gamma(\mathcal{G})$ for it. See fig.s 1,2 for the explicit correspondence.

The 2-dimensional defining representation \mathcal{Q} is obtained by regarding the group Γ as an $SU(2)$ subgroup [that is, \mathcal{Q} is the representation which acts in eq.(33)].

For the Kleinian groups Γ it is particularly important the decomposition of the tensor product of an irreducible representation D_μ with the defining 2-dimensional representation \mathcal{Q} . It is indeed at the level of this decomposition that the relation between these groups and the simply laced Dynkin diagrams is more explicit [23]. Furthermore this decomposition plays a crucial role in the explicit construction of the ALE manifolds [9]. Setting

$$\mathcal{Q} \otimes D_\mu = \bigoplus_{\nu=0}^r A_{\mu\nu} D_\nu \quad (34)$$

where D_0 denotes the identity representation, one finds that the matrix $\bar{c}_{\mu\nu} = 2\delta_{\mu\nu} - A_{\mu\nu}$ is the *extended Cartan matrix* encoded in the *extended Dynkin diagram* corresponding to the given group. Recall that the extended Dynkin diagram contains in addition to

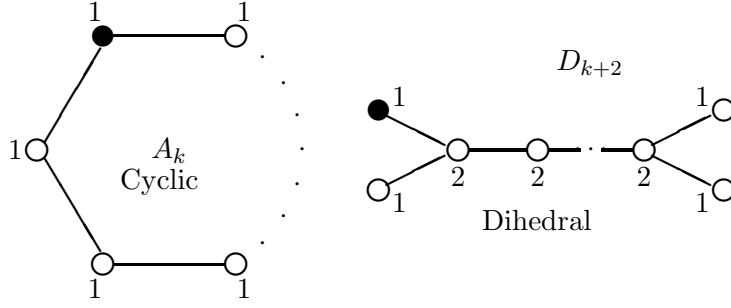


Figure 1: *Extended Dynkin diagrams of the infinite series*

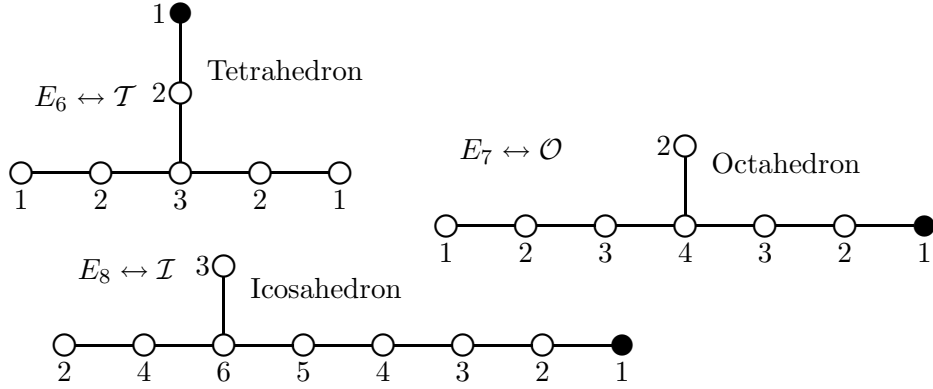


Figure 2: *Exceptional extended Dynkin diagrams*

the *dots* representing the *simple roots* $\{\alpha_1 \dots \alpha_r\}$ an *extra dot* (marked black in Figs 5, 5) representing the negative of the highest root $\alpha_0 = \sum_{i=1}^r n_i \alpha_i$ (n_i are the Coxeter numbers). There is thus a correspondence between the non-trivial conjugacy classes or equivalently the non-trivial irrepses of the group $\Gamma(\mathcal{G})$ and the simple roots of \mathcal{G} ; in this correspondence, the extended Cartan matrix provides us with the Clebsch-Gordan coefficients (34), while the Coxeter numbers n_i express the dimensions of the irreducible representations. All these informations are summarized in Figs 2,3 where the numbers n_i are attached to each of the dots; the extra dot stands for the identity representation.

•**Example** Consider the cyclic subgroups of $SU(2)$, that is the A_k -series. The defining 2-dimensional representation \mathcal{Q} is given by the matrices

$$\gamma_l \in \Gamma(A_k) \quad ; \quad \gamma_l = \mathcal{Q}_l \stackrel{\text{def}}{=} \begin{pmatrix} e^{2\pi i l / (k+1)} & 0 \\ 0 & e^{-2\pi i l / (k+1)} \end{pmatrix} \{l = 1, \dots, k\}. \quad (35)$$

It is not irreducible since all irreducible representations are one-dimensional as one sees from Fig. 2. In the j -th irreducible representation the l -th element of the group is represented by $D^{(j)}(e_l) = \nu^{jl}$, where $\nu = \exp 2\pi i / k$, $j = 1, \dots, k$. The $(k+1) \times (k+1)$ array of phases ν^{jl} appearing in the above equation is the character table. Given the \mathbb{C}^2 carrier space of the defining representation [see eq.s (33)] one can construct three algebraic invariants, namely

$$z = z_1 z_2 \quad ; \quad x = (z_1)^{k+1} \quad ; \quad y = (z_2)^{k+1} \quad (36)$$

that satisfy the polynomial relation

$$W_{A_k}(x, y, z) \stackrel{\text{def}}{=} xy - z^{k+1} = 0. \quad \square \quad (37)$$

Analogous invariants and polynomials (see table 1) can be constructed for the other kleinian subgroups.

ALE manifolds and resolution of simple singularities

The polynomial constraint $W_\Gamma(x, y, z) = 0$ plays an important role in the construction of the ALE manifolds. Indeed, as we are going to see, the vanishing locus in \mathbf{C}^3 of the *potential* $W_\Gamma(x, y, z)$ coincides with the space of equivalence classes \mathbf{C}^2/Γ , that is with the singular orbifold limit of the self-dual manifold \mathcal{M}_Γ . According to the standard procedure of deforming singularities [24, 25, 26, 27] there is a corresponding family of smooth manifolds $\mathcal{M}_\Gamma(t_1, t_2, \dots, t_r)$ obtained as the vanishing locus \mathbf{C}^3 of a *deformed potential*:

$$\tilde{W}_\Gamma(x, y, z; t_1, t_2, \dots, t_r) = W_\Gamma(x, y, z) + \sum_{\alpha=1}^r t_\alpha \mathcal{P}^{(\alpha)}(x, y, z) \quad (38)$$

where t_α are complex numbers (the moduli of the complex structure of \mathcal{M}_Γ) and $\mathcal{P}^{(\alpha)}(x, y, z)$ is a basis spanning the chiral ring

$$\mathcal{R} = \frac{\mathbf{C}[x, y, z]}{\partial W} \quad (39)$$

of polynomials in x, y, z that do not vanish upon use of the vanishing relations $\partial_x W = \partial_y W = \partial_z W = 0$. The dimension of this chiral ring $|\mathcal{R}|$ is precisely equal to the number of non-trivial conjugacy classes (or of non trivial irreducible representations) of the finite group Γ . From the geometrical point of view this implies an identification between the number of complex structure deformations of the ALE manifold and the number r of non-trivial conjugacy classes discussed above.

In this framework one can describe the homology of the manifolds $\mathcal{M}_\Gamma(t_\alpha)$ with $t \neq 0$. The non-contractible two-cycles $c_\alpha, \alpha = 1, \dots, r$ (each isomorphic to a copy of \mathbf{CP}^1) can be put into correspondence with the vertices of the non-extended Dynkin diagram for Γ . The intersection matrix of the c_α is the negative of the Cartan matrix:

$$c_\alpha \cdot c_\beta = \bar{c}_{\alpha\beta}. \quad (40)$$

The Kronheimer construction, that we shortly describe, shows that the base manifold (simply denoted as \mathcal{M}) of an ALE space is diffeomorphic to the space $\mathcal{M}_\Gamma(t_\alpha)$ supporting the resolution of the orbifold \mathbf{C}^2/Γ . Therefore the equation (40) applies to the generators of its second homology group. In particular we see that

$$\begin{aligned} \tau &= \dim H_{\mathbf{C}}^2(\mathcal{M}) = \dim H_2(\mathcal{M}) = \\ &= \text{rank of the corresponding Lie Algebra} = \\ &= \# \text{ non trivial conj. classes in } \Gamma = |\mathcal{R}|. \end{aligned} \quad (41)$$

Table 1: *KLEINIAN GROUP versus ALE MANIFOLD properties*

Γ .	$W(x, y, z)$	$\mathcal{R} = \frac{\mathcal{C}[x, y, z]}{\partial W}$	$ \mathcal{R} $	#c. c.	$\tau \equiv \chi - 1$
A_k	$xy - z^{k+1}$	$\{1, z, \dots, z^{k-1}\}$	k	$k + 1$	k
D_{k+2}	$x^2 + y^2 z + z^{k+1}$	$\{1, y, z, y^2, z^2, \dots, z^{k-1}\}$	$k + 2$	$k + 3$	$k + 2$
$E_6 = \mathcal{T}$	$x^2 + y^3 + z^4$	$\{1, y, z, yz, z^2, yz^2\}$	6	7	6
$E_7 = \mathcal{O}$	$x^2 + y^3 + yz^3$	$\{1, y, z, y^2, z^2, yz, y^2 z\}$	7	8	7
$E_8 = \mathcal{I}$	$x^2 + y^3 + z^5$	$\{1, y, z, z^2, yz, z^3, yz^2, yz^3\}$	8	9	8

where τ is the Hirzebruch signature ⁶. These results are summarized in Table 1.

Kronheimer construction

The HyperKähler quotient is performed on a suitable flat HyperKähler space \mathcal{S} that now we define. Given any finite subgroup of $SU(2)$, Γ , consider a space \mathcal{P} whose elements are two-vectors of $|\Gamma| \times |\Gamma|$ complex matrices: $p \in \mathcal{P} = (A, B)$. The action of an element $\gamma \in \Gamma$ on the points of \mathcal{P} is the following:

$$\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{\gamma} \begin{pmatrix} u_\gamma & i\bar{v}_\gamma \\ iv_\gamma & \bar{u}_\gamma \end{pmatrix} \begin{pmatrix} R(\gamma)AR(\gamma^{-1}) \\ R(\gamma)BR(\gamma^{-1}) \end{pmatrix} \quad (42)$$

where the two dimensional matrix in the r.h.s. is the realization of γ in the defining representation \mathcal{Q} of Γ , while $R(\gamma)$ is the regular, $|\Gamma|$ -dimensional representation ⁷. This transformation property identifies \mathcal{P} , from the point of view of the representations of Γ , as $\mathcal{Q} \otimes \text{End}(R)$. The space \mathcal{P} can be given a quaternionic structure, representing its elements as “quaternions of matrices”:

$$p \in \mathcal{P} = \begin{pmatrix} A & iB^\dagger \\ iB & A^\dagger \end{pmatrix} \quad A, B \in \text{End}(R) . \quad (43)$$

The space \mathcal{S} is the subspace of Γ -invariant elements in \mathcal{P} :

$$\mathcal{S} \stackrel{\text{def}}{=} \{p \in \mathcal{P} / \forall \gamma \in \Gamma, \gamma \cdot p = p\} . \quad (44)$$

Explicitly the invariance condition reads:

$$\begin{pmatrix} u_\gamma & i\bar{v}_\gamma \\ iv_\gamma & \bar{u}_\gamma \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} R(\gamma^{-1})AR(\gamma) \\ R(\gamma^{-1})BR(\gamma) \end{pmatrix} . \quad (45)$$

The space \mathcal{S} is elegantly described for all Γ 's using the associated Dynkin diagram.

⁶for ALE manifolds $\tau = \chi - 1$, χ being the Euler characteristic; it just counts the normalizable selfdual forms

⁷The basis vectors e_γ of the regular representation R are in one-to-one correspondence with the group elements γ and transform as $R(\gamma)e_\delta = e_{\gamma\delta}$, $\forall \gamma, \delta \in \Gamma$.

A two-vector of matrices can be thought of also as a matrix of two-vectors: that is, $\mathcal{P} = \mathcal{Q} \otimes \text{Hom}(R, R) = \text{Hom}(R, \mathcal{Q} \otimes R)$. Decomposing into irreducible representations the regular representation, $R = \bigoplus_{\nu=0}^r n_{\nu} D_{\nu}$, using eq.(34) and the Schur's lemma, one gets

$$\mathcal{S} = \bigoplus_{\mu, \nu} A_{\mu, \nu} \text{Hom}(\mathbf{C}^{n_{\mu}}, \mathbf{C}^{n_{\nu}}). \quad (46)$$

The dimensions of the irrepses, n_{μ} are expressed in Fig.s (5,5). From eq.(46) the real dimension of \mathcal{S} follows immediately: $\dim \mathcal{S} = \sum_{\mu, \nu} 2A_{\mu\nu} n_{\mu} n_{\nu}$ implies, recalling that $A = 2\mathbf{1} - \bar{c}$ [see eq.(34)] and that for the extended Cartan matrix $\bar{c}n = 0$,

$$\dim \mathcal{S} = 4 \sum_{\mu} n_{\mu}^2 = 4|\Gamma|. \quad (47)$$

The quaternionic structure of \mathcal{S} can be seen by simply writing its elements as in eq.(43) with A, B satisfying the invariance condition eq.(45). Then the HyperKähler forms and the metric are described by $\Theta = \text{Tr}(d\bar{m} \wedge m)$ and $ds^2 \mathbf{1} = \text{Tr}(d\bar{m} \otimes dm)$. The trace is taken over the matrices belonging to $\text{End}(R)$ in each entry of the quaternion.

•**Example** *The space \mathcal{S} can be easily described when Γ is the cyclic group A_{k-1} . The order of A_{k-1} is k ; the abstract multiplication table is that of \mathbf{Z}_k and from it we can immediately read off the matrices of the regular representation. One has $R(e_1)_{lm} = \delta_{l, m+1}$ and of course $R(e_j) = (R(e_1))^j$. Actually, the invariance condition eq.(45) is best solved by changing basis so as to diagonalize the regular representation. Let $\nu = e^{\frac{2\pi i}{k}}$, so that $\nu^k = 1$. The change of basis is performed by the matrix $S_{ij} = \frac{\nu^{-ij}}{\sqrt{k}}$; in the new basis $R(e_j) = \text{diag}(1, \nu^j, \nu^{2j}, \dots, \nu^{(k-1)j})$. The entries are the representatives of e_j in the unidimensional irrepses.*

The explicit solution of eq.(45) is given in the above basis by

$$(A)_{lm} = \delta_{l, m+1} u^l \quad ; \quad (B)_{lm} = \delta_{l, m-1} v^l \quad (48)$$

We see that these matrices are parametrized in terms of $2k$ complex, i.e. $4k = 4|A_{k-1}|$ real parameters. \square

Consider the action of $SU(|\Gamma|)$ on \mathcal{P} given, using the quaternionic notation for the elements of \mathcal{P} , by

$$\forall g \in SU(|\Gamma|), g : \begin{pmatrix} A & iB^{\dagger} \\ iB & A^{\dagger} \end{pmatrix} \mapsto \begin{pmatrix} gAg^{-1} & igB^{\dagger}g^{-1} \\ igBg^{-1} & gA^{\dagger}g^{-1} \end{pmatrix}. \quad (49)$$

It is easy to see that this action is a triholomorphic isometry of \mathcal{P} : ds^2 and Θ are invariant. Let F be the subgroup of $SU(|\Gamma|)$ which *commutes with the action of Γ on \mathcal{P}* , the Γ -action described in eq.(42). Then the action of F descends to $\mathcal{S} \subset \mathcal{P}$ to give a *triholomorphic isometry*: the metric and HyperKähler forms on \mathcal{S} are just the restriction of those on \mathcal{P} . It is therefore possible to take the HyperKähler quotient of \mathcal{S} with respect to F .

Let $\{f_A\}$ be a basis of generators for \mathcal{F} , the Lie algebra of F . Under the infinitesimal action of $f = \mathbf{1} + \lambda^A f_A \in F$, the variation of $m \in \mathcal{S}$ is $\delta m = \lambda^A \delta_A m$, with

$$\delta_A m = \begin{pmatrix} [f_A, A] & i[f_A, B^{\dagger}] \\ i[f_A, B] & [f_A, A^{\dagger}] \end{pmatrix}. \quad (50)$$

The components of the momentum map (see (7)) are then given by

$$\mathcal{D}_A = \text{Tr}(\bar{m} \delta_A m) \stackrel{\text{def}}{=} \text{Tr} \begin{pmatrix} f_A \mathcal{D}_3(m) & f_A \mathcal{D}_-(m) \\ f_A \mathcal{D}_+(m) & f_A \mathcal{D}_3(m) \end{pmatrix} \quad (51)$$

so that the real and holomorphic maps $\mathcal{D}_3 : \mathcal{S} \rightarrow \mathcal{F}^*$ and $\mathcal{D}_+ : \mathcal{S} \rightarrow \mathbf{C} \times \mathcal{F}^*$ can be represented as matrix-valued maps:⁸

$$\begin{aligned} \mathcal{D}_3(m) &= -i([A, A^\dagger] + [B, B^\dagger]) \\ \mathcal{D}_+(m) &= ([A, B]) . \end{aligned} \quad (52)$$

Let $\mathcal{Z} \equiv \mathcal{Z}^*$ be the dual of the centre of \mathcal{F} . In correspondence with a level $\zeta = \{\zeta^3, \zeta^+\} \in \mathbf{R}^3 \otimes \mathcal{Z}$ we can form the HyperKähler quotient $\mathcal{M}_\zeta \stackrel{\text{def}}{=} \mathcal{D}^{-1}(\zeta)/F$. *Varying ζ and Γ every ALE space can be obtained as \mathcal{M}_ζ .*

First of all, it is not difficult to check that \mathcal{M}_ζ is four-dimensional. As for the space \mathcal{S} , there is a nice characterization of the group F in terms of the extended Dynkin diagram associated with Γ :

$$F = \bigotimes_{\mu} U(n_{\mu}) . \quad (53)$$

One must however set the determinant of the elements to one, since $F \subset SU(|\Gamma|)$. F has a $U(n_{\mu})$ factor for each dot of the diagram, n_{μ} being associated to the dot as in Figs 1,2. F acts on the various “components” of \mathcal{S} [which are in correspondence with the edges of the diagram, see eq.(46)] as dictated by the structure diagram. From eq.(53) it is immediate to derive that $\dim F = \sum_{\mu} n_{\mu}^2 - 1 = |\Gamma| - 1$. It follows that

$$\dim \mathcal{M}_\zeta = \dim \mathcal{S} - 4 \dim F = 4|\Gamma| - 4(|\Gamma| - 1) = 4 . \quad (54)$$

•Example *The structure of F and the momentum map for its action are very simply worked out in the A_{k-1} case. An element f of F must commute with the action of A_{k-1} on \mathcal{P} , eq.(42), where the two-dimensional representation in the l.h.s. is given in eq.(35). Then f must have the form*

$$f = \text{diag}(e^{i\varphi_0}, e^{i\varphi_1}, \dots, e^{i\varphi_{k-1}}) ; \quad \sum \varphi_i = 0 . \quad (55)$$

Thus \mathcal{F} is just the algebra of diagonal traceless k -dimensional matrices, which is $k-1$ -dimensional. Choose a basis of generators for \mathcal{F} , for instance $f_1 = \text{diag}(1, -1, \dots), f_2 = \text{diag}(1, 0, -1, \dots), \dots, f_{k-1} = \text{diag}(1, 0, \dots, -1)$. From eq.(52) one gets directly the components of the momentum map:

$$\begin{aligned} \mathcal{D}_{3,A} &= |u^0|^2 - |v_0|^2 - |u^{k-1}|^2 - |v_{k-1}|^2 - |u^A|^2 - |v_A|^2 + |u^{A-1}|^2 - |v_{A-1}|^2 \\ \mathcal{D}_{+,A} &= u^0 v_0 - u^{k-1} v_{k-1} - u^A v_A + u^{A-1} v_{A-1} . \quad \square \end{aligned} \quad (56)$$

In order for \mathcal{M}_ζ to be a manifold, it is necessary that F acts freely on $\mathcal{D}^{-1}(\zeta)$. This happens or not depending on the value of $\zeta \in \mathcal{Z}$. Again, a simple characterization of \mathcal{Z} can be given in terms of the simple Lie algebra \mathcal{G} associated with Γ [9]. There exists an isomorphism between \mathcal{Z} and the Cartan subalgebra \mathcal{H} of \mathcal{G} . Thus we have

$$\dim \mathcal{Z} = \dim \mathcal{H} = \text{rank } \mathcal{G} = \# \text{ of non trivial conj. classes in } \Gamma . \quad (57)$$

⁸It is easy to see that indeed the matrices $[A, A^\dagger] + [B, B^\dagger]$ and $[A, B]$ belong to the Lie algebra of traceless matrices \mathcal{F} ; practically we identify \mathcal{F}^* with \mathcal{F} by means of the Killing metric.

The space \mathcal{M}_ζ turns out to be singular when, under the above identification $\mathcal{Z} \sim \mathcal{H}$, any of the level components $\zeta^i \in \mathbf{R}^3 \otimes \mathcal{Z}$ lies on the walls of a Weyl chamber. In particular, as the point $\zeta^i = 0$ for all i is identified with the origin in the root space *the space \mathcal{M}_0 is singular*. We will see in a moment that \mathcal{M}_0 corresponds to the *orbifold limit* \mathbf{C}^2/Γ of a family of ALE manifolds with boundary at infinity S^3/Γ .

To see that this is general, choose the natural basis $\{e_\delta\}$ for the regular representation R . Define then the space $L \subset \mathcal{S}$ as follows:

$$L = \left\{ \begin{pmatrix} C \\ D \end{pmatrix} \in \mathcal{S} / C, D \text{ are diagonal in the basis } \{e_\delta\} \right\}. \quad (58)$$

For every element $\gamma \in \Gamma$ there is a pair of numbers (c_γ, d_γ) given by the corresponding entries of C, D : $C \cdot e_\gamma = c_\gamma e_\gamma$, $D \cdot e_\gamma = d_\gamma e_\gamma$. Applying the invariance condition eq.(45), which is valid since $L \subset \mathcal{S}$, it results that

$$\begin{pmatrix} c_{\gamma \cdot \delta} \\ d_{\gamma \cdot \delta} \end{pmatrix} = \begin{pmatrix} u_\gamma & i\bar{v}_\gamma \\ iv_\gamma & \bar{u}_\gamma \end{pmatrix} \begin{pmatrix} c_\delta \\ d_\delta \end{pmatrix}. \quad (59)$$

We can identify L with \mathbf{C}^2 associating for instance $(C, D) \in L \mapsto (c_0, d_0) \in \mathbf{C}^2$. Indeed all the other pairs (c_γ, d_γ) are determined in terms of eq.(59) once (c_0, d_0) are given. By eq.(59) the action of Γ on L induces exactly the action of Γ on \mathbf{C}^2 that we considered in eq.s (32,33).

Note that we can directly realize \mathbf{C}^2/Γ as an affine algebraic surface in \mathbf{C}^3 (see eq. (37)) by expressing the coordinates x, y and z of \mathbf{C}^3 in terms of the matrices $(C, D) \in L$.

•**Example** *The explicit parametrization of the matrices in \mathcal{S} in the A_{k-1} case (which was given in eq.(48) in the basis in which the regular representation R is diagonal), can be conveniently rewritten in the “natural” basis $\{e_\gamma\}$ via the matrix S^{-1} . The subset L of diagonal matrices (C, D) is given by*

$$C = c_0 \text{diag}(1, \nu, \nu^2, \dots, \nu^{k-1}), \quad D = d_0 \text{diag}(1, \nu^{k-1}, \nu^{k-2}, \dots, \nu), \quad (60)$$

where $\nu = e^{\frac{2\pi i}{k}}$. This is nothing but the fact that $\mathbf{C}^2 \sim L$. The set of pairs $(\nu^m c_0, \nu^{k-m} d_0)$, $m = 0, 1, \dots, k-1$ is an orbit of Γ in \mathbf{C}^2 and determines the corresponding orbit of Γ in L . To describe \mathbf{C}^2/A_{k-1} we identify $(x, y, z) \in \mathbf{C}^3$, such that $xy = z^k$, as

$$x = \det C \quad ; \quad y = \det D, \quad ; \quad z = \frac{1}{k} \text{Tr} CD. \quad \square \quad (61)$$

It is quite easy to show the following fundamental fact: *each orbit of F in $\mathcal{D}^{-1}(0)$ meets L in one orbit of Γ* . Because of the above identification between L and \mathbf{C}^2 , this leads to the proof that \mathcal{M}_0 is isometric to \mathbf{C}^2/Γ .

•**Example** *Let us show explicitly in the case of the cyclic groups the one-to-one correspondence between the orbits of F in $\mathcal{D}^{-1}(0)$ and those of Γ in L . Choose the basis where R is diagonal. Then $(A, B) \in \mathcal{S}$ has the form of eq. (48). Now, the relation $xy = z^k$ (eq. (37)) holds true also when, in eq. (61), the pair $(C, D) \in L$ is replaced by an element $(A, B) \in \mathcal{D}^{-1}(0)$. Indeed the elements $(A, B) \in \mathcal{D}^{-1}(0)$ can be described solving eq. (52) at zero r.h.s.. It gives $u_j = |u_0|e^{i\phi_j}$ and $v_j = |v_0|e^{i\psi_j}$ and $\psi_j = \Phi - \phi_j \forall j$ for a certain phase Φ . Then we immediately check that such a pair $(A, B) \in \mathcal{D}^{-1}(0)$ satisfies $xy = z^k$ if $x = \det A$, $y = \det B$ and $z = (1/k) \text{Tr} AB$. We are left with $k+3$ parameters (the k phases ϕ_j , $j = 0, 1, \dots, k-1$, plus the absolute values $|u_0|$ and*

$|v_0|$ and the phase Φ). Indeed $\dim \mathcal{D}^{-1}(0) = \dim \mathcal{M} - 3 \dim F = 4|\Gamma| - 3(|\Gamma| - 1) = |\Gamma| + 3$, where $|\Gamma| = \dim \Gamma = k$.

Now we perform the quotient of $\mathcal{D}^{-1}(0)$ with respect to F . Given a set of phases f_i such that $\sum_{i=0}^{k-1} f_i = 0 \bmod 2\pi$ and given $f = \text{diag}(e^{if_0}, e^{if_1}, \dots, e^{if_{k-1}}) \in F$, the orbit of F in $\mathcal{D}^{-1}(0)$ passing through (A, B) is given by (fAf^{-1}, fBf^{-1}) . Choosing $f_j = f_0 + j\psi + \sum_{n=0}^{j-1} \phi_n$, $j = 1, \dots, k-1$, with $\psi = -\frac{1}{k} \sum_{n=0}^{k-1} \phi_n$, and f_0 determined by the condition $\sum_{i=0}^{k-1} f_i = 0 \bmod 2\pi$, one has

$$(fAf^{-1})_{lm} = a_0 \delta_{l,m+1} \quad ; \quad (fBf^{-1})_{lm} = b_0 \delta_{l,m-1} \quad (62)$$

where $a_0 = |u_0|e^{i\psi}$ and $b_0 = |v_0|e^{i(\Phi-\psi)}$. Since the phases ϕ_j are determined modulo 2π , it follows that ψ is determined modulo $\frac{2\pi}{k}$. Thus we can say $(a_0, b_0) \in \mathbf{C}^2/\Gamma$. This is the one-to-one correspondence between $\mathcal{D}^{-1}(0)/F$ and \mathbf{C}^2/Γ . \square

5 Resolution of ALE singularities $W_\Gamma(t^\alpha)$ and Fayet-Iliopoulos parameters

So far we have reviewed the main points of the Kronheimer construction. In particular we have shown the constructive definition of the quaternionic flat space \mathcal{S} and of the “gauge group” acting on it by triholomorphic isometries needed to retrieve an ALE space as a HyperKähler quotient. That is, we have described the necessary ingredients to specify, according to the procedure outlined in sec. 3, an N=4 renormalizable field theory (the *microscopic theory*) whose low-energy effective action (the *macroscopic theory*) is the sigma-model on the ALE space under consideration ⁹.

We do not insist on the mathematical proofs of the main statements of Kronheimer’s work (in particular, the identification of *all* ALE spaces with \mathcal{M}_ζ). We rather choose to illustrate, in the specific case of the cyclic subgroups, an *explicit* relation between the parameters $\zeta^i \in \mathcal{Z}$, $i = 1, 2, 3$ of the HyperKähler construction (the levels of the momentum map) and the deformation parameters t^α appearing in eq. (38). We divide the ζ parameters in r -parameters (the real levels of the \mathcal{D}^3 momentum map) and s -parameters (the complex levels of the \mathcal{D}^+ momentum map) since this was the notation utilized in sec. 3. This relation tells us explicitly which is the “deformed” potential describing an ALE space, obtained as a HyperKähler quotient with levels $\{r, s\}$, as an hypersurface in \mathbf{C}^3 . We stress that the parameters r, s are coupling parameters (the N=4 generalizations of Fayet-Iliopoulos parameters) in the “microscopic” N=4 lagrangian while the t^α are parameters in the σ -model (the “macroscopic” description), since they appear in the definition of the target space, and in particular of its complex structure. This gives a physical interest to the relation we describe.

To find the desired relation, we have in practice to find a “deformed” relation between the invariants x, y, z . To this purpose, we focus on the holomorphic part of the momentum map, i.e. on the equation $[A, B] = \Sigma_0$, where $\Sigma_0 = \text{diag}(s_0, s_1, \dots, s_{k-1})$ with $s_0 = -\sum_{i=1}^{k-1} s_i$. Recall the expression (48) for the matrices A and B . Calling $a_i = u_i v_i$, $[A, B] = \Sigma_0$ implies that $a_i = a_0 + s_i$ for $i = 1, \dots, k-1$. Now, let $\Sigma = \text{diag}(s_1, \dots, s_{k-1})$.

⁹Of course, to carry out explicitly until the end computations analogous to those for the Calabi metrics is extremely complicated; indeed the form of the metric that would result from this quotient is in general not known, with the exception of the Eguchi-Hanson case.

We have

$$xy = \det A \det B = a_0 \prod_{i=1}^{k-1} (a_0 + s_i) = a_0^k \det \left(1 + \frac{1}{a} \Sigma \right) = \sum_{i=0}^{k-1} a_0^{k-i} S_i(\Sigma). \quad (63)$$

The $S_i(\Sigma)$ are the symmetric polynomials in the eigenvalues of Σ , defined by $\det(1 + \Sigma) = \sum_{i=0}^{k-1} S_i(\Sigma)$. In particular, $S_0 = 1$ and $S_1 = \sum_{i=1}^{k-1} s_i$. Define $S_k(\Sigma) = 0$, so that $xy = \sum_{i=0}^k a_0^{k-i} S_i(\Sigma)$, and note that $z = \frac{1}{k} \text{Tr} AB = a_0 + \frac{1}{k} S_1(\Sigma)$. Then the desired deformed relation between x , y and z is obtained by substituting $a_0 = z - \frac{1}{k} S_1(\Sigma)$ in (63), obtaining finally

$$xy = \sum_{m=0}^k \sum_{n=0}^{k-m} \binom{k-m}{n} \left(-\frac{1}{k} S_1(\Sigma) \right)^{k-m-n} S_m(\Sigma) z^n = \sum_{n=0}^k t_n z^n. \quad (64)$$

$$\implies t_n = \sum_{m=0}^{k-n} \binom{k-n}{m} \left(-\frac{1}{k} S_1(\Sigma) \right)^{k-m-n} S_m(\Sigma). \quad (65)$$

Notice in particular that $t_k = 1$ and $t_{k-1} = 0$, i.e. $xy = z^k + \sum_{n=0}^{k-2} t_n z^n$, which means that the deformation proportional to z^{k-1} is absent. This establishes a clear correspondence between the momentum map construction and the polynomial ring $\mathbf{C}[x, y, z]/\partial W$ where $W(x, y, z) = xy - z^k$ [compare with eq. (38)]. Moreover, note that we have only used one of the momentum map equations, namely $[A, B] = \Sigma_0$. The equation $[A, A^\dagger] + [B, B^\dagger] = R$ has been completely ignored. This means that the deformation of the complex structure is described by the parameters Σ , while the parameters R describe the deformation of the Kähler class.

The relation (65) can also be written in a simple factorized form, namely

$$xy = \prod_{i=0}^{k-1} (z - \mu_i), \quad (66)$$

where

$$\begin{aligned} \mu_i &= \frac{1}{k} (s_1 + s_2 + \cdots + s_{i-1} - 2s_i + s_{i+1} + \cdots + s_k), \quad i = 1, \dots, k-1 \\ \mu_0 &= -\sum_{i=1}^k \mu_i = \frac{1}{k} S_1(\Sigma). \end{aligned} \quad (67)$$

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